Functional Interval Observer for Discrete-Time Nonlinear Lipschitz Systems

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Abstract: This paper considers discrete-time nonlinear Lipschitz systems with unknown but bounded disturbances. Thanks to the existence of bounding decomposition functions for mixed monotone mappings and the Lyapunov approach, we present a functional interval observer to achieve stable upper and lower bounds for a linear function of the system state. The simulation on a numerical example is given to illustrate the theoretical results.

Keywords: Functional interval observer, nonlinear Lipschitz systems, discrete-time systems, mixed monotone mappings, Lyapunov approach.

1. INTRODUCTION

In recent years, interval observers have enjoyed tremendous progress in the control community due to their simple design and promising performance to put bounds on state of systems which are affected by fluctuating uncertainties. Different from traditional observers like Luenberger observer which only give a point estimate as time tends to infinity, interval observers offer some guarantees during transient periods. Indeed, they provide us with readily usable estimate at any time instant, even when system parameters or disturbances are changing or large. Thus, they can satisfy an important requirement of estimation in application which is to monitor and detect system faults (Chevet et al. (2021)).

Under the assumptions that the initial conditions and disturbances are unknown but bounded, the idea of interval observers is to provide two observers: the first one is always higher than the true state while the second one is always lower. The construction of such an observer is directly or indirectly based on positive systems (Gouzé et al. (2000)). Some works on interval observers are devoted to various families of one-dimensional (1D) and two-dimensional (2D) linear systems (Chevet et al. (2022); Mazenc et al. (2014)), bilinear systems (Dinh and Ito (2016)), linear parameter varying (LPV) (Ellero et al. (2019)), linear switched systems (Dinh et al. (2020a)) or even to some classes of nonlinear systems with some restrictions (Raïssi et al. (2011); Dinh et al. (2014); Ito and Dinh (2020)).

In parallel, functional observers have been considered (Trinh and Fernando (2012)). In practice, the observation object usually belongs to a linear function of the state vectors (e.g., the measured part of the system state is regularly a linear function of the state). Consequently, for some use case scenarios, we only need to estimate the linear function of the state variables instead of estimating themselves. This helps to reduce the complexity and the order of the observer design. In the framework of functional interval observer design, (Gu et al. (2018)) is devoted to linear continuous-time systems, and then is extended to other classes such as discrete-time linear systems (Che et al. (2020)), multivariable linear systems (Meyer (2019)), fractional-order systems (Huong and Yen (2020)), switched descriptor systems (Huang et al. (2022)) and so on. However, To the best of our knowledge, no functional interval observer has been proposed for Lipschitz systems for which many constructions of asymptotic observers (see for instance (Dinh et al. (2015))) and interval observers (see for instance (Meyer et al. (2018); Khajenejad and Yong (2020))) have been proposed. It motivates the contribution of the present paper: *to design a functional interval observer for a class of Lipschitz systems.* By assuming the existence of bounding decomposition functions for mixed monotone mappings, a framer for a linear function of the state can be constructed. Next, a Lyapunov approach is employed to give sufficient conditions to ensure the stability of the mentioned framer.

The remainder of the present paper is organized as follows. General preliminaries are presented in Section 2. In Section 3, the proposed Lipschitz system and assumptions on the considered plant are discussed. The design procedure for the functional interval observer as well as the proposed stability conditions are introduced in Section 4. Section 5 proposes a numerical simulation, which illustrates the mathematical results. Finally, Section 6 gathers concluding remarks and perspectives.

2. PRELIMINARIES

The identity matrix is denoted by I. A null matrix 0 is a matrix all of whose entries are zero. The Euclidean norm of a vector $x \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times m}$ are respectively denoted by $\|x\|$ and $\|M\|$. Inequalities must be understood *component-wise*, i.e., for $x_a = [x_{a,1}, ..., x_{a,n}]^\top \in \mathbb{R}^n$ and $x_b = [x_{b,1}, ..., x_{b,n}]^\top \in \mathbb{R}^n$, $x_a \le x_b$ if and only if, for all $i \in \{1, ..., n\}$, $x_{a,i} \le x_{b,i}$. For a matrix $M \in \mathbb{R}^{n\times m}$, we define M^+, M^-, and the matrix of absolute values of all elements $|M|$ by $M^+ = \max(M, 0)$, $M^- = M^+ - M$, $|M| = M^+ + M^-$.

Next, we introduce some definitions and lemmas which are useful for the functional interval observer design.

Lemma 1. (Efimov et al., 2013, Section IIA) Consider vectors $x, \underline{x}, \overline{x}$ in $\mathbb{R}^n$ such that $\underline{x} \le x \le \overline{x}$ and a constant matrix $A \in \mathbb{R}^{m\times n}$, then

$$A^+\underline{x} - A^-\overline{x} \le Ax \le A^+\overline{x} - A^-\underline{x}, \tag{1}$$

with $A^+ = \max\{0, A\}$, $A^- = A^+ - A$.

Definition *1.* (Yang et al., 2019, Definition 4) A mapping $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$ is mixed monotone if there exists $\mathcal{F}_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ such that

(1) $\mathcal{F}_c(\alpha, \alpha) = \mathcal{F}(\alpha)$.
(2) For a fixed second argument, $\mathcal{F}_c$ is nondecreasing with respect to its first argument, i.e., $\alpha_1 \le \alpha_2 \Rightarrow \mathcal{F}_c(\alpha_1, \beta) \le \mathcal{F}_c(\alpha_2, \beta)$.
(3) For a fixed first argument, $\mathcal{F}_c$ is nonincreasing with respect to its second argument, i.e., $\beta_1 \le \beta_2 \Rightarrow \mathcal{F}_c(\alpha, \beta_1) \ge \mathcal{F}_c(\alpha, \beta_2)$.

The function $\mathcal{F}_c$ is called a decomposition function of $\mathcal{F}$.

Lemma 2. Let $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^m$ be mixed monotone and $\mathcal{F}_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ be a decomposition of $\mathcal{F}$. Given α, $\underline{\alpha}$, $\overline{\alpha}$ in $\mathbb{R}^n$ such that $\underline{\alpha} \le \alpha \le \overline{\alpha}$. Then,

$$\mathcal{F}_c(\underline{\alpha}, \overline{\alpha}) \le \mathcal{F}(\alpha) \le \mathcal{F}_c(\overline{\alpha}, \underline{\alpha}). \tag{2}$$

Proof. From Definition 1, the proof is obvious.

3. PROBLEM STATEMENT

Consider the nonlinear systems with bounded disturbances

$$x(k+1) = Ax(k) + \mathcal{F}(\Pi x(k)) + w(k), \tag{3}$$
$$y(k) = Cx(k) \tag{4}$$

where $A \in \mathbb{R}^{n_x\times n_x}$, $C \in \mathbb{R}^{n_y\times n_x}$, $\Pi \in \mathbb{R}^{n_g\times n_x}$ are known matrices, $x(k) \in \mathbb{R}^{n_x}$ and $y(k) \in \mathbb{R}^{n_y}$ are respectively the state and the output. The disturbance $w(k) \in \mathbb{R}^{n_x}$ is supposed unknown but bounded by known values, i.e.,

$$\underline{w} \le w(k) \le \overline{w}, \tag{5}$$

where $\overline{w}$, $\underline{w}$ are known vectors. The function $\mathcal{F} : \mathbb{R}^{n_g} \to \mathbb{R}^{n_x}$ is known globally Lipschitz. Without loss of generality, we assume that $n_x \ge n_y \ge 1$. Moreover, the initial condition at the instant $k = 0$ is assumed to be bounded by two known bounds

$$\underline{x}(0) \le x(0) \le \overline{x}(0). \tag{6}$$

Remark 1.
- All the results of this paper can be extended straightforwardly to the case where the matrix A and the function $\mathcal{F}$ depend on k explicitly as well as the input and the measurement noise are present.
- The class of nonlinear system described by (3) and (4) can cover a wide range of practical applications, especially the mechatronics system driven by multiple electric motors (vehicles, drones, propeller systems etc.). In such systems, the forces/torques on the machine's body is commonly described by a nonlinear function of the motion variables of the local motor actuators. A typical example is the electric vehicle driven by in-wheel motors (Nguyen et al. (2019); Pacejka (2005)).
- The system (3) is more general than the family of nonlinear discrete-time systems presented in (Dinh et al. (2020b)). If we choose $\Pi = C$, (3) reduces to the considered system in (Dinh et al. (2020b)).

Assumption 1. Functions $\mathcal{F}$ is mixed monotone with decomposition functions $\mathcal{F}_c$.

Assumption 2. For all h_1, h_2 in $\mathbb{R}^{n_g}$, there exists a strictly positive constant $L_{\mathcal{F}}$ such that

$$\|\mathcal{F}(h_1) - \mathcal{F}(h_2)\| \le L_{\mathcal{F}}\|h_1 - h_2\|. \tag{7}$$

$\mathcal{F}$ is then called a globally $L_{\mathcal{F}}$-Lipschitz continuous function.

Lemma 3. Let $\mathcal{F} : \mathbb{R}^{n_g} \to \mathbb{R}^{n_x}$ be globally $L_{\mathcal{F}}$-Lipschitz and mixed monotone with decomposition function $\mathcal{F}_c : \mathbb{R}^{n_g} \times \mathbb{R}^{n_g} \to \mathbb{R}^{n_x}$. Given $\underline{h} = [\underline{h}_1, ..., \underline{h}_n]^T$, $h = [h_1, ..., h_n]^T$, $\overline{h} = [\overline{h}_1, ..., \overline{h}_n]^T \in \mathbb{R}^{n_g}$ satisfying $\underline{h} \le h \le \overline{h}$. Then

$$\mathcal{F}_c(\overline{h}, \underline{h}) = \mathcal{F}(z_1) + C_{\mathcal{F}}(\overline{h} - \underline{h}), \tag{8}$$
$$\mathcal{F}_c(\underline{h}, \overline{h}) = \mathcal{F}(z_2) + C_{\mathcal{F}}(\underline{h} - \overline{h}), \tag{9}$$

with $C_{\mathcal{F}} \in \mathbb{R}^{n_x\times n_g}$ is computed in (Yang et al., 2019, Theorem 2), where $z_1 = [z_{1,1}, ..., z_{1,n}]^T$, $z_2 = [z_{2,1}, ..., z_{2,n}]^T \in \mathbb{R}^{n_g}$ and for all $i \in \{1, ..., n\}$, $z_{1,i}$ and $z_{2,i}$ are either $\overline{h}_i$ or $\underline{h}_i$ given in (Yang et al., 2019, Theorem 2).

Furthermore,

$$\|\mathcal{F}_c(\overline{h}, \underline{h}) - \mathcal{F}_c(\underline{h}, \overline{h})\| \le (L_{\mathcal{F}} + 2\|C_{\mathcal{F}}\|)\,\|\overline{h} - \underline{h}\|. \tag{10}$$

Proof. The equalities (8) and (9) are results from (Yang et al., 2019, Theorem 2). Since $\underline{h} \le h \le \overline{h}$ and for all $i \in \{1, ..., n\}$, $z_{1,i}$ and $z_{2,i}$ are either $\overline{h}_i$ or $\underline{h}_i$, then $\underline{h} \le h, z_1, z_2 \le \overline{h}$. Consequently, we obtain

$$\|z_1 - z_2\| \le \|\overline{h} - \underline{h}\|. \tag{11}$$

Combining (8) and (9) and by applying the triangle inequality we have

$$\begin{aligned} &\|\mathcal{F}_c(\overline{h}, \underline{h}) - \mathcal{F}_c(\underline{h}, \overline{h})\| \\ &= \|\mathcal{F}(z_1) - \mathcal{F}(z_2) + 2C_{\mathcal{F}}(\overline{h} - \underline{h})\| \\ &\le \|\mathcal{F}(z_1) - \mathcal{F}(z_2)\| + 2\|C_{\mathcal{F}}\|\|\overline{h} - \underline{h}\|. \end{aligned}$$

Because $\mathcal{F}$ is globally $L_{\mathcal{F}}$-Lipschitz continuous, from Assumption 2 we dedcue that

$$\|\mathcal{F}_c(\overline{h}, \underline{h}) - \mathcal{F}_c(\underline{h}, \overline{h})\| \le (L_{\mathcal{F}} + 2\|C_{\mathcal{F}}\|)\,\|\overline{h} - \underline{h}\|.$$

Thanks to (11), we can conclude (10).

Definition *2.* The following dynamics for all $k \ge 0$

$$\begin{cases} \overline{g}(k+1) = \overline{\mathcal{G}}(\overline{g}(k), y(k), \mathcal{F}_c(\overline{h}(k), \underline{h}(k)), \overline{w}, \underline{w}), \\ \underline{g}(k+1) = \underline{\mathcal{G}}(\underline{g}(k), y(k), \mathcal{F}_c(\underline{h}(k), \overline{h}(k)), \overline{w}, \underline{w}), \end{cases} \tag{12}$$

associated with the initial conditions $\overline{g}(0) = \overline{\mathcal{G}}_0(\overline{x}(0), \underline{x}(0))$, $\underline{g}(0) = \underline{\mathcal{G}}_0(\overline{x}(0), \underline{x}(0))$, and with outputs

$$\begin{cases}\overline{h}(k)=\overline{\mathcal{H}}(\overline{g}(k),\underline{g}(k),y(k)),\forall k>0,\\ \underline{h}(k)=\underline{\mathcal{H}}(\overline{g}(k),\underline{g}(k),y(k)),\forall k>0,\\ \overline{h}(0)=\overline{\mathcal{H}}_0(\overline{x}(0),\underline{x}(0)),\\ \underline{h}(0)=\underline{\mathcal{H}}_0(\overline{x}(0),\underline{x}(0)),\end{cases} \tag{13}$$

where $\overline{w}$, $\underline{w}$ are defined in (5); $\underline{x}(0)$, $\overline{x}(0)$ are defined in (6); $\mathcal{F}_c$ is defined in Definition 1; $\overline{\mathcal{G}}$, $\underline{\mathcal{G}}$, $\overline{\mathcal{G}}_0$, $\underline{\mathcal{G}}_0$, $\overline{\mathcal{H}}$, $\underline{\mathcal{H}}$, $\overline{\mathcal{H}}_0$, $\underline{\mathcal{H}}_0$ are functions in $\mathbb{R}^{n_g}$, are called a *functional interval observer* for the linear function of the state of (3)

$$h(k)=\Pi x(k),\ \text{with the matrix } \Pi \text{ given in (3)} \tag{14}$$

if

(1) $\underline{h}(k)\le h(k)\le \overline{h}(k)$ for all $k>0$,
(2) $\lim_{k\to\infty}\|\overline{\mathcal{H}}(\overline{g}(k),\underline{g}(k),y(k))-\underline{\mathcal{H}}(\overline{g}(k),\underline{g}(k),y(k))\|=0$ when $w(k)=0$ for all $k\in\mathbb{N}$.

Remark 2.
- Different from the definition of interval observer in (Mazenc et al. (2014)), the functional interval observer only recovers the linear function of the state variables, which can reduce the order and complexity of the observer construction (Che et al. (2020)).
- Interestingly, if we take the specific case where $\Pi = I$, (13) becomes a *new interval observer design* for the nonlinear Lipschitz system $x(k+1)=Ax(k)+\mathcal{F}(x(k))+w(k)$. This design is totally different from existing results (Meyer et al. (2018); Khajenejad and Yong (2020)) in the literature. The specific case is simulated in the Section 5.
- For the design simplicity, we consider the same matrix Π in (3) and (14). The extension to two different matrices is not difficult and follows the same routine given in Section 4.

Goal. Given the nonlinear system (3) with the measurements (4), we design a functional interval observer which estimates the linear function of the state variables (14).

4. FUNCTIONAL INTERVAL OBSERVER DESIGN

In this section, we arrive at the above-mentioned goal by proposing two stable bounds $\overline{h}(k)$ and $\underline{h}(k)$ for the linear function of the state $x(t)$. In the theory of interval observer, we have to achieve two properties: (i) framer property which is the notion of providing intervals in which state variable stay and (ii) stability property which cares the length of estimated intervals.

Theorem 1. (Framer property). Let Assumption 1 hold. Consider the nonlinear system (3) with the measurement (4) and the dynamic extension as follows

$$\overline{g}(k+1)=D\overline{g}(k)+Jy(k)+R^+\mathcal{F}_c(\overline{h}(k),\underline{h}(k)) - R^-\mathcal{F}_c(\underline{h}(k),\overline{h}(k))+R^+\overline{w}-R^-\underline{w},\forall k\ge 0, \tag{15}$$

$$\underline{g}(k+1)=D\underline{g}(k)+Jy(k)+R^+\mathcal{F}_c(\underline{h}(k),\overline{h}(k)) - R^-\mathcal{F}_c(\overline{h}(k),\underline{h}(k))+R^+\underline{w}-R^-\overline{w},\forall k\ge 0, \tag{16}$$

$$\overline{h}(k)=S^+\overline{g}(k)-S^-\underline{g}(k)+Ny(k),\forall k>0, \tag{17}$$

$$\underline{h}(k)=S^+\underline{g}(k)-S^-\overline{g}(k)+Ny(k),\forall k>0, \tag{18}$$

associated with the initial conditions

$$\overline{g}(0)=R^+\overline{x}(0)-R^-\underline{x}(0), \tag{19}$$

$$\underline{g}(0)=R^+\underline{x}(0)-R^-\overline{x}(0), \tag{20}$$

$$\overline{h}(0)=\Pi^+\overline{x}(0)-\Pi^-\underline{x}(0), \tag{21}$$

$$\underline{h}(0)=\Pi^+\underline{x}(0)-\Pi^-\overline{x}(0), \tag{22}$$

where $\overline{w}$, $\underline{w}$, $\overline{x}(0)$, $\underline{x}(0)$ and Π are respectively defined in (5), (6) and (14), $D\in\mathbb{R}^{n_g\times n_g}$, $J\in\mathbb{R}^{n_g\times n_y}$, $R\in\mathbb{R}^{n_g\times n_x}$, $S\in\mathbb{R}^{n_g\times n_g}$, $N\in\mathbb{R}^{n_g\times n_y}$ are design parameters to be determined. If (5) and (6) hold, D is a nonnegative matrix, and the following equations

$$RA-DR=JC, \tag{23}$$

$$\Pi=SR+NC, \tag{24}$$

are satisfied, then (17)-(18) are a framer for (3) satisfying $\underline{h}(k)\le h(k)=\Pi x(k)\le\overline{h}(k)$ for all $k\ge 0$.

Remark 3. Conditions (23) and (24) are not selective (Che et al., 2020, Lemma 5-6):

- Given a nonnegative matrix $D\in\mathbb{R}^{n_g\times n_g}$ whose eigenvalues are different from those of A in (3) and a matrix $J\in\mathbb{R}^{n_g\times n_y}$, there always exists a unique matrix $R\in\mathbb{R}^{n_g\times n_x}$ satisfying the Sylvester equation (23).
- Given two arbitrary matrices $\Pi\in\mathbb{R}^{n_g\times n_x}$ and $R\in\mathbb{R}^{n_g\times n_x}$. If a matrix $S\in\mathbb{R}^{n_g\times n_g}$ is chosen such that $\operatorname{rank}\begin{bmatrix}C\\ \Pi-SR\end{bmatrix}=\operatorname{rank}C=n_y$ with C defined in (4), then there always exists a matrix $N\in\mathbb{R}^{n_g\times n_y}$ satisfying the equation (24).

Proof. Let $\overline{e}_g(k)=\overline{g}(k)-Rx(k)$ and $\underline{e}_g(k)=Rx(k)-\underline{g}(k)$. Then from (3)-(4), (15), (16) and bearing in mind (14), we have

$$\overline{e}_g(k+1)=D\overline{g}(k)+JCx(k)-RAx(k) + \overline{\mathcal{F}}_c(\overline{h}(k),\underline{h}(k))-R\mathcal{F}(h(k))+\overline{W}(k), \tag{25}$$

$$\underline{e}_g(k+1)=RAx(k)-D\underline{g}(k)-JCx(k) + R\mathcal{F}(h(k))-\underline{\mathcal{F}}_c(\overline{h}(k),\underline{h}(k))+\underline{W}(k), \tag{26}$$

where

$$\overline{\mathcal{F}}_c(\overline{h}(k),\underline{h}(k))=R^+\mathcal{F}_c(\overline{h}(k),\underline{h}(k))-R^-\mathcal{F}_c(\underline{h}(k),\overline{h}(k)), \tag{27}$$

$$\underline{\mathcal{F}}_c(\overline{h}(k),\underline{h}(k))=R^+\mathcal{F}_c(\underline{h}(k),\overline{h}(k))-R^-\mathcal{F}_c(\overline{h}(k),\underline{h}(k)), \tag{28}$$

$$\overline{W}(k)=R^+\overline{w}-R^-\underline{w}-Rw(k), \tag{29}$$

$$\underline{W}(k)=Rw(k)-(R^+\underline{w}-R^-\overline{w}). \tag{30}$$

Thanks to (23), it follows that

$$\overline{e}_g(k+1)=D\overline{e}_g(k)+\overline{\mathcal{F}}_c(\overline{h}(k),\underline{h}(k))-R\mathcal{F}(h(k))+\overline{W}(k), \tag{31}$$

$$\underline{e}_g(k+1)=D\underline{e}_g(k)+R\mathcal{F}(h(k))-\underline{\mathcal{F}}_c(\overline{h}(k),\underline{h}(k))+\underline{W}(k). \tag{32}$$

Let $\overline{e}_h(k)=\overline{h}(k)-h(k)$, $\underline{e}_h(k)=h(k)-\underline{h}(k)$. Now, we prove by induction that for all $k\ge 0$, $\overline{e}_g(k)\ge 0$, $\underline{e}_g(k)\ge 0$, $\overline{e}_h(k)\ge 0$, $\underline{e}_h(k)\ge 0$. According to (6), (19), (20), (21), (22) and Lemma 1, the property is satisfied at the instant 0, i.e, $\overline{e}_g(0)\ge 0$, $\underline{e}_g(0)\ge 0$, $\overline{e}_h(0)\ge 0$, $\underline{e}_h(0)\ge 0$. Assume that it is satisfied at the step $k>0$. Lemma 2 and the fact that $\underline{h}(k)<h(k)<\overline{h}(k)$ imply that $\mathcal{F}_c(\underline{h}(k),\overline{h}(k))\le\mathcal{F}(h(k))\le\mathcal{F}_c(\overline{h}(k),\underline{h}(k))$. So, by using Lemma 1, one can easily deduce that $\underline{\mathcal{F}}_c(\overline{h}(k),\underline{h}(k))\le$

$R\mathcal{F}(h(k)) \leq \overline{\mathcal{F}}_c(\overline{h}(k), \underline{h}(k))$. From (5) and Lemma 1, we have $\overline{W}(k) \geq 0$ and $\underline{W}(k) \geq 0$ for all $k \geq 0$. Additionally, the matrix D is nonnegative. Thus, from (31) and (32), it follows that $\overline{e}_g(k+1) \geq 0$ and $\underline{e}_g(k+1) \geq 0$. Hence,

$$\underline{g}(k+1) \leq Rx(k+1) \leq \overline{g}(k+1). \tag{33}$$

By left multiplying (34) by nonnegative matrices S^+ and S^- and bearing in mind (17), (18), we get

$$\begin{aligned} \underline{h}(k+1) - Ny(k+1) &\leq SRx(k+1) && (34)\\ &\leq \overline{h}(k+1) - Ny(k+1). && (35) \end{aligned}$$

According to (24), $SR = \Pi - NC$. Therefore,

$$\underline{h}(k+1) \leq \Pi x(k+1) \leq \overline{h}(k+1). \tag{36}$$

Consequently, $\overline{e}_h(k+1) \geq 0$, $\overline{e}_h(k+1) \geq 0$ and the induction assumption is satisfied at the step $k+1$.

Theorem 2. (Stability property). If Assumptions 1-2 are satisfied and there exist a positive definite matrix P and a positive constant $\varepsilon_{\mathcal{F}}$ such that

$$\begin{aligned} &D^T\left(P + \varepsilon_{\mathcal{F}}^{-1} P|R||R|^T P\right) D - P \\ &+ \left(\varepsilon_{\mathcal{F}} + \|R\|^2\|P\|\right) L_{c\mathcal{F}}^2 \|S\|^2 I \preceq 0, \end{aligned} \tag{37}$$

with $L_{c\mathcal{F}} = L_{\mathcal{F}} + 2\|C_{\mathcal{F}}\|$ where $L_{\mathcal{F}}$ and $C_{\mathcal{F}}$ are respectively defined in Assumption 2 and Lemma 3. Then, the framer proposed in (17)-(18) becomes functional interval observer for the linear function $\Pi x(k)$ of the state of (3).

Proof. From (17) and (18), we have for all $k > 0$,

$$\overline{h}(k) - \underline{h}(k) = |S|\left(\overline{g}(k) - \underline{g}(k)\right). \tag{38}$$

Remark that $\lim_{k\to\infty} \|\overline{g}(k) - \underline{g}(k)\| = 0 \Leftrightarrow \lim_{k\to\infty} \|\overline{h}(k) - \underline{h}(k)\| = 0$. Therefore, to prove the stability of the framer (17)-(18), we need to prove that if the condition (37) holds, then when $w(k) = 0$ for all $k \in \mathbb{N}$, $\lim_{k\to\infty} \|\overline{g}(k) - \underline{g}(k)\| = 0$. Let $\Delta(k) = \overline{g}(k) - \underline{g}(k)$. Using (15) and (16), when $w(k) = 0$ for all $k \in \mathbb{N}$, $\Delta(k+1)$ can be rewritten as

$$\begin{aligned} \Delta(k+1) &= D\Delta(k) \\ &\quad + |R|\left[F_c(\overline{h}(k), \underline{h}(k)) - F_c(\underline{h}(k), \overline{h}(k))\right]. \end{aligned} \tag{39}$$

Lyapunov approach (Khalil (2002)). We consider a Lyapunov function candidate $V(k) = \Delta(k)^T P \Delta(k)$, where $P \succ 0$ and the goal is to find a condition such that $V(k+1) - V(k) < 0$.

Consider the system (39), we have

$$\begin{aligned} V(k+1) - V(k) &= \Delta^T(k)(D^T PD - P)\Delta(k) \\ &\quad + 2\Delta^T(k) D^T P|R|\left[\mathcal{F}_c(\overline{h}(k), \underline{h}(k)) - \mathcal{F}_c(\underline{h}(k), \overline{h}(k))\right] \\ &\quad + \left[\mathcal{F}_c(\overline{h}(k), \underline{h}(k)) - \mathcal{F}_c(\underline{h}(k), \overline{h}(k))\right]^T |R|^T P|R| \\ &\quad \times \left[\mathcal{F}_c(\overline{h}(k), \underline{h}(k)) - \mathcal{F}_c(\underline{h}(k), \overline{h}(k))\right]. \end{aligned}$$

Thanks to Lemma 3, a positive constant $L_{c\mathcal{F}} = L_{\mathcal{F}} + 2||C||_{\mathcal{F}}$ can be computed such that

$$\begin{aligned} ||\mathcal{F}_c(\overline{h}(k), \underline{h}(k)) - \mathcal{F}_c(\underline{h}(k), \overline{h}(k))|| &\leq L_{c\mathcal{F}} ||\overline{h}(k) - \underline{h}(k)|| \\ &= L_{c\mathcal{F}} \|S\| \|\Delta(k)\|. \end{aligned}$$

It follows that for any constant $\varepsilon_{\mathcal{F}} > 0$,

$$\begin{aligned} V(k+1) - V(k) \leq& \Delta^T(k)(D^T PD - P)\Delta(k) \\ &+ \varepsilon_{\mathcal{F}}^{-1} \Delta^T(k) D^T P|R||R|^T PD\Delta(k) \\ &+ \left[\mathcal{F}_c(\overline{h}(k), \underline{h}(k)) - \mathcal{F}_c(\underline{h}(k), \overline{h}(k))\right]^T \\ &\times \left[\mathcal{F}_c(\overline{h}(k), \underline{h}(k)) - \mathcal{F}_c(\underline{h}(k), \overline{h}(k))\right] \\ &\times \left(\varepsilon_{\mathcal{F}} + \|R\|^2\|P\|\right) \\ \leq& \Delta^T(k)(D^T PD - P)\Delta(k) \\ &+ \varepsilon_{\mathcal{F}}^{-1} \Delta^T(k) D^T P|R||R|^T PD\Delta(k) \\ &+ \left(\varepsilon_{\mathcal{F}} + \|R\|^2\|P\|\right) L_{c\mathcal{F}}^2 \|S\|^2 \|\Delta(k)\|^2. \end{aligned}$$

Based on the above analysis, $V(k+1) - V(k) < 0$ if (37) holds.

5. NUMERICAL EXAMPLE

We apply Theorem 1-2 to the nonlinear Lipschitz system

$$x(k+1) = \frac{1}{4}x(k) + \frac{1}{8}\sin\left(\frac{1}{4}x(k)\right) + w(k), \tag{40}$$

$$y(k) = x(k), \tag{41}$$

with $x(k) \in \mathbb{R}$, $C = 1$, $A = \Pi = \frac{1}{4}$ and $\mathcal{F}(\Pi x(k)) = \frac{1}{8}\sin\left(\frac{1}{4}x(k)\right)$, where $w(k) = \frac{1}{4}\cos(2k)$ represents disturbances.

To verify Assumption 1, we select the function $\mathcal{F}_c : \mathbb{R}^2 \to \mathbb{R}$

$$\mathcal{F}_c(a, b) = \frac{1}{8}a + \frac{1}{8}\left(\sin\left(\frac{1}{4}b\right) - b\right), \tag{42}$$

which is such that

$$\mathcal{F}_c(a, a) = \mathcal{F}(a), \quad \forall a \in \mathbb{R}, \tag{43}$$

and is nondecreasing with respect to the variable a and nonincreasing with respect to the variable b.

Choosing $S = \frac{1}{8}$, $N = \frac{1}{8}$, $R = 1$, $D = \frac{1}{8} > 0$ and $J = \frac{1}{8}$, the conditions (23) and (24) are satisfied.

Next, we verify the stability condition (37). We deduce from (42) that for all $\overline{h}, \underline{h} \in \mathbb{R}, \mathcal{F}_c(\overline{h}, \underline{h}) = \frac{1}{8}(\overline{h} - \underline{h}) + \frac{1}{8}\sin\left(\frac{1}{4}\underline{h}\right)$ so $C_{\mathcal{F}} = \frac{1}{8}$. From the numerical example (40), we have $L_{\mathcal{F}} = \frac{1}{32}$. By using the positive definitive quadratic function $V(k) = x^2(k)$ (i.e., $P = 1$) and choosing $\varepsilon_{\mathcal{F}} = \frac{1}{2}$, one can easily conclude that $D^T\left(P + \varepsilon_{\mathcal{F}}^{-1} P|R||R|^T P\right) D - P + \left(\varepsilon_{\mathcal{F}} + \|R\|^2\|P\|\right)(L_f + 2C_f)^2 \|S\|^2 I < 0$.

We present the following simulation with the initial condition $h(0) = \frac{1}{4}x(0) = \frac{1}{4} \times 1 = \frac{1}{4}$, $\overline{x}(0) = 1.5$, $\underline{x}(0) = 0.5$, $\overline{h}(0) = \frac{1}{4} \times 1.5 = 0.375$, $\underline{h}(0) = \frac{1}{4} \times 0.5 = 0.125$, and $\overline{w} = -\underline{w} = \frac{1}{4}$. Figure 1 gives the linear function $h = \Pi x$ of the solution x and the bounds provided by the functional interval observer (17)-(18).

The simulation confirms the mathematical result: the linear function $h(k)$ lies in the upper bound $\overline{h}(k)$ and the lower bound $\underline{h}(k)$ in the presence of disturbance $w(k)$.

Specific case where $\Pi = I$. The system (40) becomes

$$x(k+1) = \frac{1}{4}x(k) + \frac{1}{8}\sin(x(k)) + w(k). \tag{44}$$

Choosing $S = \frac{1}{8}$, $N = \frac{7}{8}$, $R = 1$, $D = \frac{1}{8} > 0$ and $J = \frac{1}{8}$, *a new interval observer design* for the system (44) which

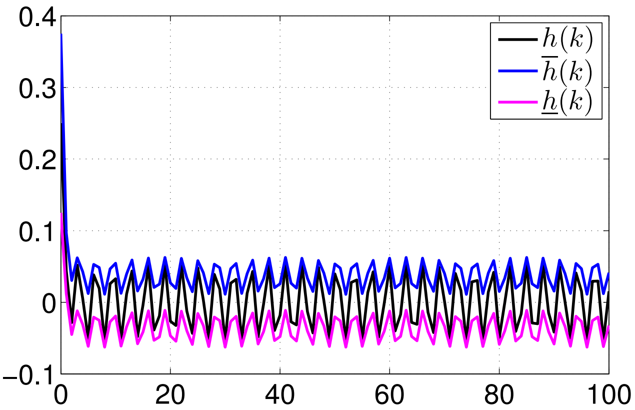

Fig. 1. Functional interval observer (17)-(18) for the numerical example (40)-(41).

is different from existing results in the literature can be proposed as follows

$$\overline{g}(k+1) = \frac{1}{8}\left[\overline{g}(k) + y(k) + \overline{h}(k) + \left(\sin\left(\underline{h}(k)\right) - \underline{h}(k)\right)\right] + \overline{w}, \quad \forall k \geq 0, \tag{45}$$

$$\overline{g}(k+1) = \frac{1}{8}\left[\underline{g}(k) + y(k) + \underline{h}(k) + \left(\sin\left(\overline{h}(k)\right) - \overline{h}(k)\right)\right] + \underline{w}, \quad \forall k \geq 0, \tag{46}$$

$$\overline{x}(k) = \frac{1}{8}\overline{g}(k) + \frac{7}{8}y(k), \quad \forall k > 0, \tag{47}$$

$$\underline{x}(k) = \frac{1}{8}\underline{g}(k) + \frac{7}{8}y(k), \quad \forall k > 0. \tag{48}$$

Similarly to the general case above, the stability condition (37) is satisfied. We present the following simulation with the initial condition $x(0) = 1$, $\overline{x}(0) = 1.5$ $\underline{x}(0) = 0.5$, and $\overline{w} = -\underline{w} = \frac{1}{4}$. Figure 2 gives the solution x of (44) and the bounds provided by the interval observer (47)-(48).

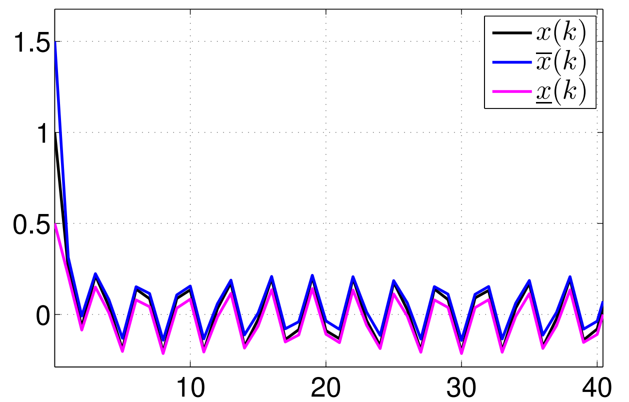

Fig. 2. Interval observer (47)-(48) for the numerical example (44)-(41).

For this specific case, the solution $x(k)$ is framed between the tight interval $[\underline{x}(k)\ \overline{x}(k)]$. We can see that the accuracy of the proposed interval observer is very good in the presence of disturbance $w(k)$.

6. CONCLUSION

The main contribution of this paper is to design a functional interval observer for a linear function of the state of a nonlinear Lipschitz system affected by unknown but bounded disturbances. Thanks to a decomposition function of the mixed monotone mapping and Lyapunov approach, the observer is provided and the interval error is proved to be input-to-state stable. For future work, we will extend the result for a more general class of nonlinear systems and will consider unknown inputs.

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