B-Spline Parametrized Solution of Robust PID Control using the Generalized KYP Lemma

Masato Kanematsu*, Gijs Hilhorst, Hiroshi Fujimoto, and Goele Pipeleers

***Abstract*— This paper presents a novel open-loop shaping approach for robust PID controller design, relying on polynomial spline parameterizations. The proposed approach exploits the generalized Kalman Yakubovic Popov (KYP) lemma to derive parameter-dependent linear matrix inequalities (LMIs) for robust PID synthesis. Multiple finite frequency domain specifications are taken into account to intuitively design practical controllers. By assuming piecewise polynomial parametrizations for the parameter-dependent optimization variables, and subsequently applying B-spline based relaxations, tractable conditions are derived that guarantee feasibility of the parameter-dependent LMIs for all uncertain parameter values. An elegant and effective approach results that solves the robust PID synthesis problem with limited conservatism. Numerical results demonstrate the potential of our approach.**

I. INTRODUCTION

Due to the extensive developments in modern and robust control theory over the last decades, control synthesis based on loop shaping has become semi- or fully-automated [1], [2]. However, $\mathscr{H}_\infty$ synthesis based on the KYP lemma is not widely applied in practice, since it is difficult to consider frequency-domain inequalities (FDIs) in finite frequency range directly. The common workaround is to introduce weighting functions to emphasize particular frequency ranges, but this complicates the design and generally increases the order of the resulting controller. Various approaches have been proposed to overcome these issues[3], [4], [5], [6], [7].

The generalized KYP lemma directly addresses FDIs on a finite-frequency range [8], [9], providing an elegant alternative to conventional $\mathscr{H}_\infty$ synthesis. Specifically for robust PID controller synthesis, a convex formulation for open-loop shaping can be derived with the generalized KYP lemma. However, despite its convexity, this formulation comprises parameter-dependent LMIs that should hold for an infinite number of parameter values, yielding a numerically intractable optimization problem. So-called LMI relaxations provide an elegant solution to obtain a finite set of sufficient LMIs.

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In this paper, we apply the recently developed B-spline based relaxation approach [10] to derive numerically tractable conditions for robust PID synthesis using the generalized KYP lemma. We show that approximate solutions to the robust synthesis problems can be obtained with limited conservatism. Numerical results verify that our approach is superior to the conventional approach.

The paper is organized as follows. First, the problem is formulated and polynomial splines are defined in Section II. Then, our main results are disscussed in Section III, followed by numerical validations in Section IV. Finally, the conclusions and future work are provided in Section V.

Notation

The set of nonnegative integers is denoted by $\mathbb{N}$, while $\mathbb{R}^n$ ($\mathbb{R}^{m\times n}$) is the set of real vectors (matrices) of dimension n ($m \times n$). The transpose of a matrix X is written as X^T, and complex conjugate transpose as X^*. The inequality $X \succeq 0$ ($X \preceq 0$) means that X is positive (negative) semidefinite. The sets of real positive definite (complex Hermitian) matrices of dimension $n\times n$ are denoted by $\mathbb{S}^n_+$ ($\mathbb{H}^n$). The symbol $\otimes$ indicates the Kronecker matrix product. The generalized time axis $\mathbb{T}$ equals $\mathbb{R}_+$ in continuous time and $\mathbb{N}$ in discrete time, while the operator δ denotes the time derivative $\delta f = df/dt$ in continuous time and the forward shift operator $\delta f(t) = f(t+1)$ in discrete time.

II. PROBLEM FORMULATION

We consider the uncertain linear system

$$G(\sigma) : \begin{cases} \delta x = A(\sigma)x + B(\sigma)u, \quad x(0) = 0, \\ y = C(\sigma)x + D(\sigma)u, \end{cases} \tag{1}$$

with state $x : \mathbb{T} \to \mathbb{R}^n$, input $u : \mathbb{T} \to \mathbb{R}$ and output $y : \mathbb{T} \to \mathbb{R}$. All system matrices are assumed to be real-valued and to have a piecewise polynomial dependency on the uncertain constant parameter σ. The parameter σ is confined to the closed and bounded interval $[\underline{\sigma}, \overline{\sigma}]$.

Defining a reference $r : \mathbb{T} \to \mathbb{R}$ for the output y, and a corresponding error signal $e = r - y$, we are interested in the design of a dynamic output feedback controller

$$K(\rho) : \begin{cases} \delta x_c = A_c x_c + B_c(\rho)e, \quad x_c(0) = 0, \\ u = C_c x_c + D_c(\rho)e, \end{cases} \tag{2}$$

with $x_c : \mathbb{T} \to \mathbb{R}^q$. We assume that the matrices A_c and C_c are given, while $B_c(\rho)$ and $D_c(\rho)$ depend affinely on the optimization variable $\rho \in \mathbb{R}^m$. Although the imposed parametrization is restrictive, it does include practical controllers like, for instance, PID controllers.

The objective is to optimize ρ such that the closed-loop feedback interconnection is stable and satisfies multiple finite frequency range specifications, for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$. To achieve this, we impose specifications on the open-loop transfer function $L(\lambda;\sigma,\rho) := G(\lambda;\sigma)K(\lambda;\rho)$, where $G(\lambda;\sigma)$ and $K(\lambda;\rho)$ respectively denote the transfer functions of $G(\sigma)$ and $K(\rho)$ for fixed values of σ and ρ. The open-loop transfer function is expressed in state-space form as

$$L(\sigma,\rho): \begin{cases} \delta\tilde{x} = \mathscr{A}(\sigma)\tilde{x} + \mathscr{B}(\sigma,\rho)e, \\ y = \mathscr{C}(\sigma)\tilde{x} + \mathscr{D}(\sigma,\rho)e, \end{cases}$$

with state vector $\tilde{x} = \begin{bmatrix} x^T & x_c^T \end{bmatrix}^T \in \mathbb{R}^{n+q}$. The associated state-space matrices are given by

$$\begin{bmatrix} \mathscr{A}(\sigma) & \mathscr{B}(\sigma,\rho) \\ \mathscr{C}(\sigma) & \mathscr{D}(\sigma,\rho) \end{bmatrix} = \left[\begin{array}{cc|c} A(\sigma) & B(\sigma)C_c & B_c(\rho) \\ 0 & A_c & B(\sigma)D_c(\rho) \\ \hline C(\sigma) & D(\sigma)C_c & D(\sigma)D_c(\rho) \end{array}\right].$$

To derive tractable conditions with limited conservatism for the open-loop shaping of a robust controller (2) for the uncertain system (1), we consider polynomial spline parameterizations, which are defined next.

A. Polynomial splines

Consider a scalar parameter σ on a closed and bounded interval $[\underline{\sigma}, \overline{\sigma}] \subset \mathbb{R}$, and let $\xi = (\xi_0, \ldots, \xi_{l+1})$ be a sequence of points satisfying

$$\underline{\sigma} = \xi_0 < \xi_1 < \cdots < \xi_l < \xi_{l+1} = \overline{\sigma}.$$

Then, a matrix $S(\sigma)$ is a polynomial spline (i.e., piecewise polynomial) of degree g with internal break points $\xi_1, \ldots, \xi_l$ and continuity conditions $\nu_1, \ldots, \nu_l$ if there exist polynomial matrices $P_0(\sigma), \ldots, P_l(\sigma)$ of degree g such that

$$\begin{aligned} S(\sigma) &= P_i(\sigma), \quad \text{for } \sigma \in [\xi_i, \xi_{i+1}), \quad i = 0, \ldots, l-1, \\ S(\sigma) &= P_l(\sigma), \quad \text{for } \sigma \in [\xi_l, \xi_{l+1}], \end{aligned}$$

and

$$\left.\frac{d^{j-1}P_{i-1}}{d\sigma^{j-1}}\right|_{\sigma=\xi_i} = \left.\frac{d^{j-1}P_i}{d\sigma^{j-1}}\right|_{\sigma=\xi_i}, \quad \text{for} \quad \begin{array}{c} j = 1, \ldots, \nu_i, \\ i = 1, \ldots, l. \end{array}$$

Fig. 1 illustrates this concept for a scalar-valued function.

By virtue of the Curry-Schoenberg theorem [11], $S(\sigma)$ can always be expressed in terms of particular normalized (scalar) B-spline basis functions, by considering the knot sequence

$$\lambda = (\underbrace{\xi_0, \ldots, \xi_0}_{g+1}, \underbrace{\xi_1, \ldots, \xi_1}_{g+1-\nu_1}, \ldots, \underbrace{\xi_l, \ldots, \xi_l}_{g+1-\nu_l}, \underbrace{\xi_{l+1}, \ldots, \xi_{l+1}}_{g+1}).$$

Denoting the i^{th} normalized B-spline basis function of degree g for the knot sequence $\lambda \in \mathbb{R}^{n_\lambda}$ by $B_{i,g,\lambda}(\sigma)$, the parameter-dependent matrix $S(\sigma)$ is expressed as

$$S(\sigma) = \sum_{i=1}^{n_\lambda - g - 1} C_i B_{i,g,\lambda}(\sigma), \tag{3}$$

where C_i, $i = 1, \ldots, n_\lambda - g - 1$ are matrix-valued coefficients. Specifically, for a given knot sequence λ, the B-splines

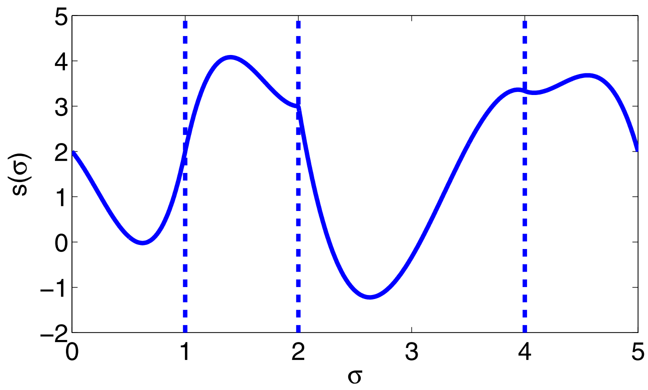

Fig. 1. A polynomial spline is a piecewise polynomial with continuity requirements. A scalar univariate polynomial spline $s: [0,5] \to \mathbb{R}$ of degree 3 with internal break points $(\xi_1, \xi_2, \xi_3) = (1,2,4)$ and continuity requirements $(\nu_1, \nu_2, \nu_3) = (2,1,2)$ is shown.

$B_{i,g,\lambda}(\sigma)$, $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, are computed using the Cox-de Boor recursive formula (see [11], p. 90):

$$B_{i,g,\lambda}(\sigma) = \frac{\sigma - \lambda_i}{\lambda_{i+g} - \lambda_i} B_{i,g-1,\lambda}(\sigma) + \frac{\lambda_{i+g+1} - \sigma}{\lambda_{i+g+1} - \lambda_{i+1}} B_{i+1,g-1,\lambda}(\sigma),$$

starting with

$$B_{i,0,\lambda}(\sigma) = \begin{cases} 1 & \text{if } \sigma \in [\lambda_i, \lambda_{i+1}), \\ 0 & \text{else.} \end{cases}$$

B-splines are commonly used as basis functions for polynomial splines, since they possess various useful properties [11], [12]:

- Positivity:

$$B_{i,g,\lambda}(\sigma) \geq 0, \quad \forall \sigma \in [\underline{\sigma}, \overline{\sigma}], \quad i = 1, \ldots, n_\lambda - g - 1.$$

- Partition of unity:

$$\sum_{i=1}^{n_\lambda - g - 1} B_{i,g,\lambda}(\sigma) = 1, \quad \forall \sigma \in [\underline{\sigma}, \overline{\sigma}].$$

- Local support:

$$\begin{aligned} B_{i,g,\lambda}(\sigma) &> 0, \quad \forall \sigma \in (\lambda_i, \lambda_{i+g+1}), \\ B_{i,g,\lambda}(\sigma) &= 0, \quad \forall \sigma \notin [\lambda_i, \lambda_{i+g+1}]. \end{aligned}$$

III. MAIN RESULTS

This section presents a novel B-spline based approach to design practical robust controllers using the generalized KYP lemma. First, we apply the generalized KYP lemma to relate finite frequency range specifications on the open-loop system to a specific parameter-dependent LMI problem. Subsequently, we exploit B-spline parameterizations to derive a tractable set of sufficient LMIs that guarantee a solution of the original parameter-dependent LMI problem.

A. Robust synthesis using the generalized KYP lemma

A unified characterization of finite frequency regions for continuous and discrete time is given by the set

$$\Lambda(\Phi,\Psi) := \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{bmatrix}\lambda \\ 1\end{bmatrix}^* \Phi \begin{bmatrix}\lambda \\ 1\end{bmatrix} = 0,\ \begin{bmatrix}\lambda \\ 1\end{bmatrix}^* \Psi \begin{bmatrix}\lambda \\ 1\end{bmatrix} \geq 0 \right\}, \tag{4}$$

where $\Phi \in \mathbb{H}^2$ and $\Psi \in \mathbb{H}^2$. Various finite frequency ranges are expressed by properly selecting the matrices Φ and Ψ, see [9] for a detailed overview.

For a given $\Pi \in \mathbb{S}^2$, an FDI on the open-loop transfer function $L(\lambda;\sigma,\rho)$ is expressed as

$$\begin{bmatrix} L(\lambda;\sigma,\rho) & I \end{bmatrix} \Pi \begin{bmatrix} L(\lambda;\sigma,\rho) & I \end{bmatrix}^* \leq 0, \tag{5}$$

and should hold for all $\lambda \in \Lambda(\Phi,\Psi)$ and all $\sigma \in [\underline{\sigma},\overline{\sigma}]$. This type of FDI can bound $L(\lambda;\sigma,\rho)$ to a large class of convex regions in the complex plane, including half planes, circles, ellipsoids and parabolas. Using the generalized KYP lemma [8], the FDI (5) is translated to the set of parameter-dependent LMIs presented in Theorem 1.

Theorem 1 (Parameter-dependent synthesis LMIs): Let Π be subdivided as

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \tag{6}$$

with $\Pi_{11} \in \mathbb{R}_+$. Then, the FDI (5) holds for all $\lambda \in \Lambda(\Phi,\Psi)$ and all $\sigma \in [\underline{\sigma},\overline{\sigma}]$ if, and only if, there exist parameter-dependent matrices $P(\sigma) \in \mathbb{H}^2$ and $Q(\sigma) \in \mathbb{H}^2$, for $\sigma \in [\underline{\sigma},\overline{\sigma}]$, such that the parameter-dependent LMIs

$$Q(\sigma) \succeq 0 \tag{7}$$

and

$$\underbrace{\begin{bmatrix} W(P(\sigma),Q(\sigma),\sigma) + V(\sigma,\rho) & \begin{bmatrix}\mathscr{B}(\sigma,\rho) \\ \mathscr{D}(\sigma,\rho)\end{bmatrix}\Pi_{11} \\ \Pi_{11}\begin{bmatrix}\mathscr{B}(\sigma,\rho) \\ \mathscr{D}(\sigma,\rho)\end{bmatrix}^* & -\Pi_{11} \end{bmatrix}}_{Z(P(\sigma),Q(\sigma),\sigma,\rho)} \preceq 0 \tag{8}$$

hold for all $\sigma \in [\underline{\sigma},\overline{\sigma}]$, where

$$W(P(\sigma),Q(\sigma),\sigma) := \begin{bmatrix} \mathscr{A}(\sigma) & I \\ \mathscr{C}(\sigma) & 0 \end{bmatrix} \left(\Phi^T \otimes P(\sigma) + \Psi^T \otimes Q(\sigma)\right) \begin{bmatrix} \mathscr{A}(\sigma) & I \\ \mathscr{C}(\sigma) & 0 \end{bmatrix}^*,$$

and

$$V(\sigma,\rho) := \begin{bmatrix} 0 & \mathscr{B}(\sigma,\rho)\Pi_{12} \\ [\mathscr{B}(\sigma,\rho)\Pi_{12}]^* & \mathrm{He}[\mathscr{D}(\sigma,\rho)\Pi_{12}] + \Pi_{22} \end{bmatrix}.$$

It should be emphasized that the affine dependency of $\mathscr{B}(\sigma,\rho)$ and $\mathscr{D}(\sigma,\rho)$ on $\rho \in \mathbb{R}^m$ is necessary for convexity of the parameter-dependent inequality (8).

To account for multiple performance specifications, we define a matrix (6) and parameter-dependent LMIs (7) and (8) for each specification $j \in \{1,\ldots,N\}$, and consider the following optimization problem:

$$\begin{aligned} &\underset{P_j(\cdot),Q_j(\cdot),\rho,\gamma_k}{\text{minimize}} && \gamma_k \\ &\text{subject to}: && Q_j(\sigma) \succeq 0, \\ & && Z_j(P_j(\sigma),Q_j(\sigma),\sigma,\rho,\gamma_j) \preceq 0, \quad \forall\sigma \in [\underline{\sigma},\overline{\sigma}], \\ & && j = 1,\ldots,N. \end{aligned} \tag{9}$$

The positive scalars γ_j, $j = 1,\ldots,N$ are performance bounds, which appear linearly in the matrix (6) corresponding to each performance specification (see, for instance, [9]). One of these bounds, γ_k, is optimized, while the remaining bounds γ_j, $j \neq k$, are fixed.

B. LMI relaxations with B-splines

In this subsection, we present a novel approach to derive a numerically tractable (i.e., finite) set of LMIs which, when feasible, guarantees feasibility of (8) for all $\sigma \in [\underline{\sigma},\overline{\sigma}]$. This approach consists of the following two steps:

1) We impose a polynomial spline parameterization on $P(\sigma)$ and $Q(\sigma)$, resulting in a finite number of optimization variables.
2) An LMI relaxation is applied on the parameter-dependent matrices $Q(\sigma)$ and $Z(P(\sigma),Q(\sigma),\sigma,\rho)$ by exploiting positivity of B-splines.

Applying step 1 yields a polynomial spline dependency of the parameter-dependent LMI $Z(P(\sigma),Q(\sigma),\sigma,\rho)$ on $\sigma \in [\underline{\sigma},\overline{\sigma}]$, which is thus conveniently expressed in terms of B-splines, as in (3). Subsequently, the positivity property of B-splines reveals that

$$\begin{aligned} &C_i \preceq 0, \quad i = 1,\ldots,n_\lambda - g - 1, \\ &\qquad \Rightarrow \quad Z(P(\sigma),Q(\sigma),\sigma,\rho) \preceq 0, \quad \forall\sigma \in [\underline{\sigma},\overline{\sigma}]. \end{aligned}$$

Hence, step 2 can be addressed by imposing positive, respectively, negative definiteness on all the B-spline coefficients of $Q(\alpha)$ and $Z(P(\sigma),Q(\sigma),\sigma,\rho)$. Generally speaking, imposing positive (negative) definiteness on all the B-spline coefficients of a polynomial spline is sufficient for positive (negative) definiteness of the polynomial spline itself.

Although a finite set of sufficient conditions for the parameter-dependent LMIs $Q(\sigma)$ and $Z(P(\sigma),Q(\sigma),\sigma,\rho)$ is readily derived from the associated B-spline coefficients, a less conservative but larger set of sufficient LMIs can be obtained by extending the B-spline bases of the parameter-dependent LMIs. Consider the univariate polynomial spline $S(\sigma)$, defined in (3). It is always possible to express $S(\sigma)$ in terms of a higher dimensional B-spline basis, by applying either degree elevation or knot insertion. Then, the spline $S(\sigma)$ can be explicitly written as

$$S(\sigma) = \sum_{i=1}^{n_{\tilde{\lambda}}-(g+d)-1} \tilde{C}_i B_{i,g+d,\tilde{\lambda}}(\sigma),$$

where $\tilde{\lambda} \supset \lambda$ and $d \in \mathbb{N}$. Requiring $\tilde{C}_i \succ 0$, $i = 1,\ldots,n_{\tilde{\lambda}} - (g+d) - 1$ is less conservative than $C_i \succ 0$, $i = 1,\ldots,n_\lambda - g - 1$, at the expense of more coefficients.

1) Degree elevation: Increasing the degree of $S(\sigma)$ by $d \in \mathbb{N}$ on each subinterval $[\xi_i, \xi_{i+1})$ corresponds to the extended knot sequence

$$\tilde{\lambda} = (\underbrace{\xi_0, \dots, \xi_0}_{g+d+1}, \underbrace{\xi_1, \dots, \xi_1}_{g+d+1-\nu_1}, \dots, \underbrace{\xi_l, \dots, \xi_l}_{g+d+1-\nu_l}, \underbrace{\xi_{l+1}, \dots, \xi_{l+1}}_{g+d+1}),$$

with $n_{\tilde{\lambda}} = n_\lambda + d(l+2)$, resulting in $d(l+1)$ more coefficients.

2) Knot insertion: Let $\tilde{\lambda}$ be constructed from λ by adding a single knot λ_{add} between two breakpoints. Then, $n_{\tilde{\lambda}} = n_\lambda + 1$, and the coefficients $\tilde{C}_i$, $i = 1, \dots, n_{\tilde{\lambda}} - g - 1$ are related to the original coefficients of (3) as follows [11]:

$$\tilde{C}_i = (1 - \beta_{i,g}(\lambda_{\text{add}}))C_{i-1} + \beta_{i,g}(\lambda_{\text{add}})C_i,$$

where the function $\beta_{i,g}$ is defined by

$$\beta_{i,g}(\sigma) = \begin{cases} 0 & \text{if} \quad \sigma \le \lambda_i, \\ \frac{x-\lambda_i}{\lambda_{i+g}-\lambda_i} & \text{if} \quad \lambda_i < \sigma < \lambda_{i+g}, \\ 1 & \text{if} \quad \lambda_{i+g} \le \sigma. \end{cases}$$

Additional reductions of conservatism can be achieved by increasing the polynomial degrees of $Q(\sigma)$ and $Z(P(\sigma), Q(\sigma), \sigma, \rho)$, or by a proper extension of their knot sequences (e.g., equidistant spacing).

IV. NUMERICAL VALIDATION

This section demonstrates the merits of B-spline parameterizations for practical robust output feedback control design, by means of numerical comparisons with existing approaches. The LMIs are implemented and solved in Matlab with the open source software package [15], which relies on Yalmip [13], and MOSEK [14].

Note that, in general, the parameter-dependent LMI (8) is complex-valued. For implementation purposes, this complex LMI is converted to a real-valued LMI by using the following equivalence:

$$X + jY \succeq 0 \Leftrightarrow \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \succeq 0,$$

where X and Y are real-valued square matrices representing the real, respectively, imaginary parts of the complex LMI $X + jY$ [16].

We consider two uncertain systems described by the continuous-time transfer functions

$$G_1(s;\sigma) = \frac{10}{(s+1)(s^2+(1+\sigma)s+10)},$$
$$G_2(s;\sigma) = \frac{10}{(s+1+\sigma)(s^2+s+10)},$$

with $\sigma \in [-0.3, 0.3]$. The objective is to design a continuous-time PID controller

$$K(s;\rho) := k_p + \frac{k_i}{s} + \frac{k_d s}{1+T_d s} \tag{10}$$

for each system, where the design parameter is selected as $\rho = \begin{bmatrix} k_p & k_i & k_d \end{bmatrix}^T$, while $T_d = 0.05$ is fixed. The PID controller (10) is expressed in the state-space form (2) with

$$\begin{bmatrix} A_c & B_c(\rho) \\ C_c & D_c(\rho) \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 0 & \frac{k_i}{T_d} \\ 1 & -\frac{1}{T_d} & k_i - \frac{k_d}{T_d^2} \\ \hline 0 & 1 & k_p + \frac{k_d}{T_d} \end{array}\right].$$

TABLE I

COMPARISON OF OUR B-SPLINE BASED APPROACH WITH THE APPROACH [9].

System	Approach	k_p	k_i	k_d	γ_2
$G_1(\sigma)$	[9]	0.2133	1.0842	0.1579	0.9373
	B-splines	0.2119	1.0743	0.1538	0.9235
$G_2(\sigma)$	[9]	0.6271	1.3021	0.2331	1.8048
	B-splines	0.5006	1.2586	0.2052	1.4309

We impose the following $N = 3$ open-loop specifications to achieve the desired closed-loop performance:

1) To ensure sensitivity reduction in the low-frequency range, we require
$$\Im[L(j\omega;\sigma)] < \gamma_1, \tag{11}$$
for all $0.05 \le \omega \le 0.45$ and all $\sigma \in [-0.3, 0.3]$.
2) To guarantee a stability margin in the middle-frequency range, we set
$$-3\Re[L(j\omega;\sigma)] + \Im[L(j\omega;\sigma)] < \gamma_2 \tag{12}$$
for all $0.05 \le \omega \le 100$ and all $\sigma \in [-0.3, 0.3]$.
3) Robust stability for high frequencies is achieved by setting
$$|L(j\omega;\sigma)| < \gamma_3, \tag{13}$$
for all $|\omega| \ge 7$ and all $\sigma \in [-0.3, 0.3]$.

The corresponding optimization problems are expressed in the form (9), where the performance bound γ_2 is minimized to optimize the stability margin, subject to the fixed bounds $\gamma_1 = -2$ and $\gamma_3 = 0.1$. To solve the parameter-dependent LMI problem (9), we select a B-spline parameterization with polynomial degree $g = 3$, $l = 40$ internal break points and $\nu_i = 3$ continuity requirements, $i = 1, \dots, l$, for all the optimization variables. Subsequently, positivity (negativity) is imposed on all the coefficients of the resulting parameterized LMIs $Q_j(\sigma)$ $(Z_j(P_j(\sigma), Q_j(\sigma), \sigma, \rho, \gamma_j))$, $j = 1, \dots, 3$. No degree elevation or knot insertion is applied.

Table I compares our B-spline based approach with the conventional approach [9]. For both approaches, the PID gains and the minimized performance bound γ_2 corresponding to $G_1(\sigma)$ and $G_2(\sigma)$ are provided. Our B-spline approach yields less conservative bounds γ_2 compared to [9] for both systems, showing improvements of 1.5%, respectively, 21%.

Additionally, Fig. 2 (Fig. 3) shows the Nyquist plots of the open-loop transfer functions corresponding to $G_1(\sigma)$ $(G_2(\sigma))$, evaluated for 11 equidistant grid points $\sigma \in$

$[-0.3, 0.3]$. The red, blue and magenta dashed lines respectively represent the FDIs (11), (12) and (13). Moreover, the red stars and magenta diamonds indicate the open-loop transfer functions evaluated at the frequency $\omega = 0.45$, respectively, $\omega = 7$. A measure of conservatism is given by the distance from the red stars (and magenta diamonds) to the dashed red (dashed magenta) line. Especially for the second example this distance is significantly reduced by using B-spline parameterizations instead of the approach [9]. Note that only the constraints (11) and (12) are active. This means that there is a trade-off between the robust stability margin and the robust sensitivity reduction, whereas condition (13) is naturally satisfied by the roll-of of the system.

V. CONCLUSIONS

This paper has presented a novel solution approach to the robust PID synthesis problem, which is a practically important controller design. The controller design specifications were translated into parameter-dependent LMIs by means of the generalized KYP lemma. A piecewise polynomial parametrization was adopted for the parameter-dependent matrix variables arising in the parameter-dependent LMIs, and B-spline based relaxations were applied to guarantee feasibility of these LMIs for all uncertain parameter values. This resulted is a numerically effective approach that solves the robust PID synthesis problem with limited conservatism. Numerical results have demonstrated that the presented approach outperforms the conventional design procedure.

A. Future Work

Generalized KYP lemma based controller synthesis enables us to design robust PID controllers satisfying multiple finite frequency specifications. By assuming B-Spline parametrizations of the parameter-dependent LMI variables, the conservatism of the robust PID synthesis is largely eliminated.

It is worth mentioning that a gain-scheduled PID controller can be obtained in a similar manner by solving a problem of the form

$$\begin{aligned} &\underset{P(\cdot),Q(\cdot),\rho(\cdot),\gamma_k(\cdot)}{\text{minimize}} && \int_{\underline{\sigma}}^{\overline{\sigma}} \gamma_k(\sigma) d\sigma \\ &\text{subject to :} && Q_j(\sigma) \succeq 0, \\ & && Z_j(P(\sigma), Q(\sigma), \sigma, \rho(\sigma), \gamma_j(\sigma)) \preceq 0, \end{aligned}$$

for all $\sigma \in [\underline{\sigma}, \overline{\sigma}]$, $j = 1, \ldots, N$. When a polynomial spline parameterization of the optimization variables is assumed, this problem can be rendered tractable by means of the B-spline based relaxations, as discussed in Section III. This gain scheduled PID design will be elaborated in future work, and compared to the robust PID design.

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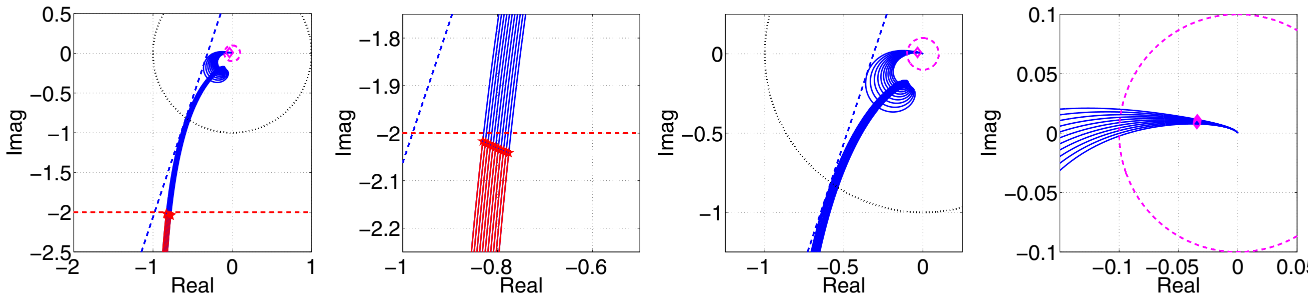
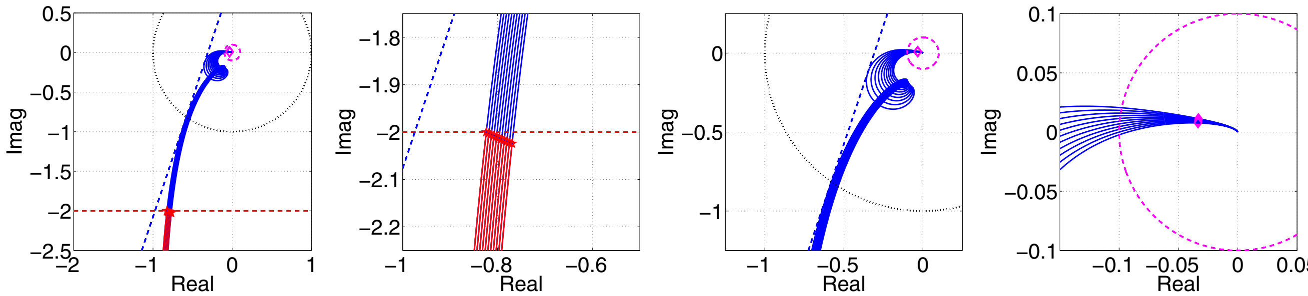
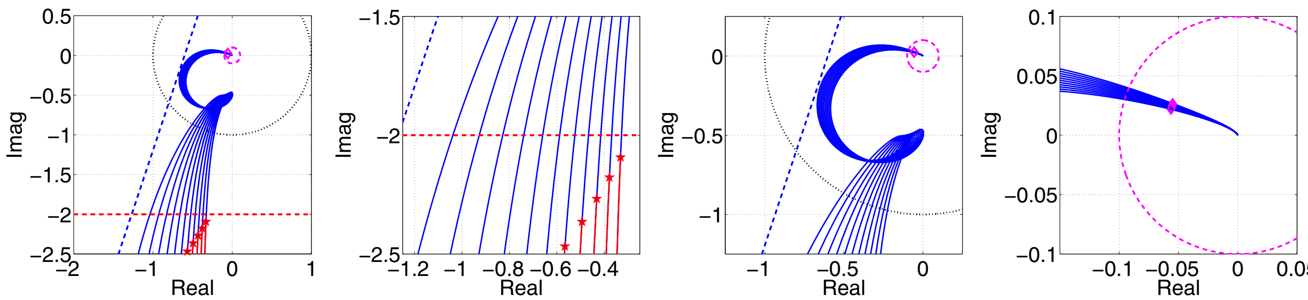


(a) Approach [9]

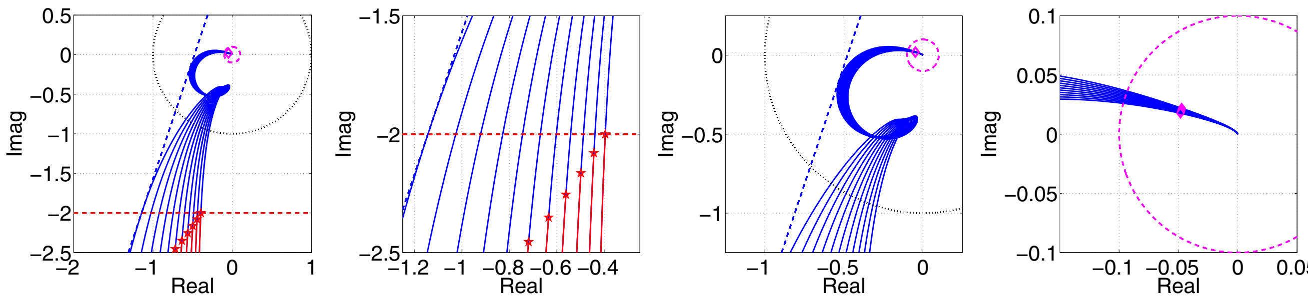


(b) B-spline based approach

Fig. 2. Nyquist diagram of the open-loop system corresponding to $G_1(\sigma)$ evaluated at 11 equidistant values for $\sigma \in [-0.3, 0.3]$. From left to right: overview, zoom around low-, middle- respectively high-frequency region.



(a) Approach [9]



(b) B-spline based approach

Fig. 3. Nyquist diagram of the open-loop system corresponding to $G_2(\sigma)$ evaluated at 11 equidistant values for $\sigma \in [-0.3, 0.3]$. From left to right: overview, zoom around low-, middle- respectively high-frequency region.